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Conformally Symmetric Anisotropic Magnetospheres in General Relativity

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A class of exact analytical solutions of Einstein–Maxwell equations is obtained for static spheres of Maugin's anisotropic magnetofluid where the space-time geometry is assumed to admit a nonstatic conformal symmetry. These solutions are found by utilizing special physical considerations.

1. INTRODUCTION

Einstein's field equations of gravitation are highly nonlinear differential equations of second order. Therefore it becomes necessary to impose some restrictions either on geometrical symmetries or on the dynamical system in obtaining exact solutions of these equations. The conditions known as isometry, self-similarity, and conformal symmetry have been utilized in the theoretical development of the subject.

In this paper, an attempt is made to integrate the Einstein–Maxwell equations for spherically symmetric and static distributions of matter possessing conformal symmetry. If the vector field $\bar{\mathscr{E}}$ is the generator of this conformal symmetry, then the space-time metric g_{ab} is mapped conformally onto itself along the trajectories of $\bar{\mathscr{E}}$, i.e.,

$$\underset{\bar{\mathscr{E}}}{\mathscr{L}} g_{ab} = \psi g_{ab} \tag{1.1}$$

where $\mathscr{L}$ is the Lie derivative operator and $\psi = \psi(x^a)$ is the conformal factor.

The essentially geometric conditions (1.1) also have a physical support as a generalization of self-similarity in hydrodynamics ($\psi = \text{const}$). Some self-similar solutions for special choices of ψ and perfect fluids have been

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extensively studied by Cahill and Taub (1971), Eardley (1974), Henriksen and Wesson (1978), and Bicnell and Henriksen (1978*a*,*b*).

Here we obtain solutions by using condition (1.1) assuming that $\bar{\mathscr{E}}$ is not only symmetric, but also nonstatic, and using a static value of ψ.

The recent theoretical work by Ruderman (1972) and Canuto (1973) on more relativistic equations of state and stellar models has suggested the introduction of anisotropic matter. Several solutions have been found by Bower and Liang (1974), Herrera and Ponce de Leon (1985), and Ponce de Leon (1987*a*,*b*) by applying different methods corresponding to static anisotropic spheres. We use various ansätze to find a class of static anisotropic spheres. The analytical solutions presented are physically reasonable and well behaved in the interior of a star.

The paper is organized as follows. In Section 2 we give the general conventions and the field equations. In Section 3 we integrate the field equations by considering conformal symmetry. Here we obtain a class of solutions for nonstatic $\bar{\mathscr{E}}$ and static ψ. This class contains flat space-time. A model for locally isotropic matter distribution and for anisotropic fluids is displayed in Section 4. A case of vanishing tangential pressure and an idealized case of incompressibility (i.e., we assume that the energy density is constant) is dealt with in Section 5. In this section we also exhibit three solutions for the matter distributions using different equations of state.

2. A SYSTEM OF FIELD EQUATIONS

In Schwarzschild coordinates the line element for static, spherically symmetric space-time can be written as

$$ds^2 = -e^{\lambda}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + e^{\nu}\,dt^2 \tag{2.1}$$

with

$$x^a = (r, \theta, \phi, t), \qquad \lambda = \lambda(r), \qquad \nu = \nu(r)$$

We consider the total energy-momentum tensor T_{ab} as the sum of the matter tensor M_{ab} and the anisotropic pressure (stress) tensor π_{ab}, i.e.,

$$T_{ab} = M_{ab} + \pi_{ab} \tag{2.2}$$

The matter tensor given by Maugin (1972) is used, which is obtained as a generalization of Lichnerowicz's (1967) scheme using the action principle. This scheme encompasses the effects of magnetization and polarization of the electromagnetic field on the internal structure of a relativistic magnetofluid. The form of the stress energy tensor for this magnetofluid is taken as

$$M_{ab} = (\rho + p + 2m)u_a u_b - (p + 2m - m\mu)g_{ab} - \mu H_a H_b \tag{2.3}$$

where ρ is the matter energy density, p is the isotropic pressure of the fluid, μ is the constant magnetic permeability, and u^a is the four-velocity of the fluid such that

$$u^a u_a = 1 \tag{2.4}$$

Here H^a is the spacelike magnetic field vector satisfying the relations

$$H^a H_a = -h^2, \qquad u^a H_a = 0 \tag{2.5}$$

and

$$2m = \mu h^2 \tag{2.6}$$

The Einstein–Maxwell equations for the magnetofluid under investigation are

$$R_{ab} = \tfrac{1}{2} R g_{ab} = K T_{ab} \tag{2.7}$$

$$(u^a H^b - u^b H^a)_{;b} = 0 \tag{2.8}$$

For the choice of comoving coordinate system, the four-velocity of the fluid can be taken as

$$u^a = (0, 0, 0, e^{-\nu/2}) \tag{2.9}$$

Hence, the space-time admitting spherical symmetry with orthogonality condition between u^a and H^a gives rise to

$$H^a = (0, 0, 0, H^1)$$

so that the Maxwell equations (2.8) yield

$$h^2 = -H_1 H^1 = \frac{N^2}{r^4} \tag{2.10}$$

and

$$m = \frac{\mu N^2}{2r^4} \tag{2.11}$$

where N is an arbitrary integration constant.

Due to the chosen symmetry, the energy density ρ, isotropic pressure p, and stress tensor π_{ab} take the form (Maartens and Maharaj, 1990)

$$\rho = \rho(r) \tag{2.12}$$

$$p = \tfrac{1}{3}[p_R(r) + 2p_T(r)] \tag{2.13}$$

$$\pi_{ab} = (P_R - P_T)(n_a n_b - \tfrac{1}{3} h_{ab}) \tag{2.14}$$

where n^a is a unit radial vector such that

$$n^a = e^{-\lambda/2}\delta_1^a \tag{2.15}$$

P_R is the radial pressure, and $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor.
Clearly,

$$H^a = \frac{N}{r^2} n^a \tag{2.16}$$

By obtaining componentwise evaluations, we derive the field equations (2.7) in explicit form as follows:

$$K(P_R - m\mu) = -r^{-2}\, e^{-\lambda}(r\nu' + 1) + r^{-2} \tag{2.17}$$

$$K(P_T + 2m - m\mu) = -\tfrac{1}{4}\, e^{-\lambda}[2\nu'' + r^{-1}(\nu' - \lambda')(2 + r\nu')] \tag{2.18}$$

$$K(\rho + m\mu) = -r^{-2}\, e^{-\lambda}(r\lambda' - 1) - r^{-2} \tag{2.19}$$

Here the value of m is given by equation (2.11).

3. A GROUP OF CONFORMAL MOTIONS AND ASSOCIATED SOLUTIONS

One method to solve the field equations (2.17)-(2.19) is to assume that the fluid space-time is mapped conformally onto itself along the direction $\bar{\varepsilon}$, so that by (1.1) the necessary conditions are

$$g_{ab,c}\mathscr{E}^c + g_{cb}\mathscr{E}^c{}_{,a} + g_{ac}\mathscr{E}^c{}_{,b} = \psi g_{ab} \tag{3.1}$$

We take the vector field $\bar{\mathscr{E}}$ in the particular form

$$\bar{\mathscr{E}} = \alpha(r, t)\delta_r + \beta(r, t)\delta_t \tag{3.2}$$

Further, we assume that the conformal factor is static, i.e.,

$$\psi = \psi(r) \tag{3.3}$$

By utilizing equations (2.1), (3.2), and (3.3) in equation (3.1), we get

$$\beta = A + \tfrac{1}{2}Bt \tag{3.4}$$

$$\alpha = \tfrac{1}{2}Cr\, e^{-\lambda/2} \tag{3.5}$$

$$\psi = C\, e^{-\lambda/2} \tag{3.6}$$

$$e^{\nu} = D^2 r^2 \exp\left(-\frac{2B}{C}\int \frac{e^{\lambda/2}}{r}\, dr\right) \tag{3.7}$$

where A, B, C, and D are arbitrary constants.

Thus, we have

$$\bar{\mathscr{E}} = \tfrac{1}{2}\psi r\delta_r + (A + \tfrac{1}{2}Bt)\delta_t \tag{3.8}$$

This generalizes the form of the isotropic conformal vector $r\delta_r + t\delta_t$ of Minkowski space-time.

The line element (2.1), using equations (3.6) and (3.7), yields

$$ds^2 = \frac{C^2}{\psi^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 D^2 r^2 \exp\left(-2B \int \frac{dr}{r\psi}\right) dt^2 \tag{3.9}$$

This solution (3.9) represents a class of spherically symmetric static space-time models for the universe filled with a Maugin anisotropic magnetofluid admitting conformal symmetry.

Comments on Metric Form (3.9):

1. The solutions obtained by Herrera *et al.* (1984) and Aherkar and Asgekar (1990) belong to the class $B = 0$.

2. Also, we can generate some magnetospheres following a group of homothetic motions by considering $\psi = \text{const}$,

$$ds^2 = E\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + D^2 r^{2-2B/\psi} dt^2 \tag{3.10}$$

where E is a constant.

This solution (3.10) is a generalization of the Tolman solution (Wainwright, 1985), since the self-similar Tolman solution can be obtained by putting $A = 0$, $B = (2-l)/bl$, $D = 1$, $E = b^2$ in (3.10). Also, we find that the solution obtained by Maartens and Maharaj (1990) for $\Delta = 0 = \psi'$ can be matched with the solution (3.10) by putting $E = 1/\psi^2$.

Note. For any choice of ψ the expressions for the matter density, radial and tangential pressures, and magnitude of the magnetic field can respectively be obtained from equations (2.11), (2.17), (2.18), and (2.19) as

$$K\rho = \frac{\psi^2}{C^2 r^2} + \frac{2\psi\psi'}{C^2 r} - \frac{1}{r^2} - \frac{K\mu^2 N^2}{2r^4} \tag{3.11}$$

$$KP_R = \frac{3\psi^2}{C^2 r^2} - \frac{2B\psi}{C^2 r^2} - \frac{1}{r^2} + \frac{K\mu^2 N^2}{2r^4} \tag{3.12}$$

$$KP_T = \frac{\psi^2}{C^2 r^2} + \frac{2\psi\psi'}{C^2 r} - \frac{2B\psi}{C^2 r^2} + \frac{B^2}{C^2 r^2} \frac{K\mu^2 N^2}{2r^4} - \frac{K\mu N^2}{r^4} \tag{3.13}$$

$$h^2 = \frac{N^2}{r^4}$$

From the definitions of the parameters associated with the timelike unit congruences u^a as given by Greenberg (1970), it is clear that for the model (3.9), the flow is expansion-free, nonshearing, and irrotational, while the acceleration is given by

$$\dot{u}^2 = \frac{(\psi - B)^2}{-C^2 r^2} \tag{3.14}$$

Also, we claim that the relative anisotropy for the model vanishes (i.e., $\sigma^2/\rho = 0$).

In order to determine the unknown functions ρ, P_R, P_T, and ψ, it is necessary to introduce additional assumptions. Instead of keeping the function ψ arbitrary, we shall find some solutions of the field equations under the constraints of physically meaningful assumptions in further sections.

4. MEASURE OF PRESSURE ANISOTROPY

It is always useful to define the measure of pressure anisotropy as

$$\Delta = K(P_T - P_R) \tag{4.1}$$

From equations (3.12) and (3.13) we obtain

$$\Delta = K(P_T - P_R) = \frac{1}{C^2 r}\left(2\psi\psi' - \frac{2}{r}\psi^2 + \frac{B^2 + C^2}{r} - \frac{\mu K N^2 C^2}{r^3}\right) \tag{4.2}$$

For the sake of simplicity we consider

$$\Delta = K(P_T - P_R) = rf'(r) \tag{4.3}$$

where $f(r)$ is an arbitrary function of r.

Equations (4.2) and (4.3) imply

$$2\psi\psi' - \frac{2}{r}\psi^2 = r^2 C^2 f'(r) - \frac{B^2 + C^2}{r} + \frac{\mu K N^2 C^2}{r^3} \tag{4.4}$$

These can be solved under standard methods to obtain the value of ψ^2 as

$$\psi^2 = r^2[C^2 f(r) + C_1] + \frac{B^2 + C^2}{2} - \frac{\mu K N^2 C^2}{4r^2} \tag{4.5}$$

where C_1 is a constant of integration.

For $C = 0 = C_1$, we get the solution representing homothetic spheres with special anisotropy (4.3). For $f(r) = 0$, that is, for a locally isotropic magnetofluid, we get the solution as

$$\psi^2 = C_1 r^2 + \frac{B^2 + C^2}{2} - \frac{\mu K N^2 C^2}{4r^2} \tag{4.6}$$

This matches with the solution obtained by Aherkar and Asgekar (1990). Clearly (4.6) gives a solution for homothetic spheres with a locally isotropic magnetofluid when $C = 0 = C_1$. Also, a model of constant anisotropy can be obtained in the form

$$\psi^2 = r^2(C^2 C_2 \log r + C_3 C^2 + C_1) + \frac{B^2 + C^2}{2} - \frac{\mu K N^2 C^2}{4r^2} \tag{4.7}$$

by substituting $f(r) = C_2 \log r + C_3$, where C_2 and C_3 are constants.

The solution (4.7) under the restrictions $C = 0 = C_1$ gives homothetic spheres with constant anisotropy.

5. A CLASS OF SOLUTIONS WITH PHYSICAL RESTRICTIONS

5.1. Vanishing Tangential Pressures

Let us consider the case $P_T = 0$. Hence, equation (3.13) with $P_T = 0$ is easily integrated by taking $B = 0$,

$$\psi^2 = \frac{C_4}{r} - \frac{\mu K N^2 C^2}{r^2}\left(1 - \frac{\mu}{2}\right) \tag{5.1}$$

where C_4 is a constant of integration.

5.2. Uniform Energy Density

The simplest form of the matter distribution throughout the interior is the uniform density.

Integrate equation (3.11) with $\rho = \text{const} = C_5$, we find

$$\psi^2 = C^2 = \frac{C^2}{3} C_5 r^2 + \frac{C_6}{r} - \frac{\mu^2 N^2 C^2 K}{2r^2} \tag{5.2}$$

where C_6 is a constant of integration.

5.3. Equation of State

The well-known equation of state is

$$P_R(\gamma - 1)\rho \tag{5.3}$$

According to (3.11) and (3.12), different solutions may be obtained by specifying the choice of γ.

5.3.1. Dust-Filled Universe

The value of ψ for a dust-filled universe (i.e., $P_R = 0$) is given as

$$\psi^2 = C_7 - C^2 - \frac{\mu^2 N^2 C^2 K}{3r^2} \tag{5.4}$$

where C_7 is a constant of integration.

5.3.2. A Radiating Universe

For a radiating universe ($\rho = 3P_R$), we find

$$\psi^2 = C_7 - \tfrac{2}{3}C^2 - \frac{2\mu^2 N^2 C^2 K}{5r^2} \tag{5.5}$$

5.3.3. Super-Dust Model

For the value $\gamma=2$, equation (5.3) gives us a super-dust model, for which we obtain

$$\psi^2 = C_7 - \frac{\mu^2 N^2 K C^2}{2r^2} \tag{5.6}$$

Remark. The relations between various arbitrary constants can be obtained from the initial and boundary conditions imposed on the system so as to make it consistent.

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